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LETTER TO THE EDITOR

On the Lie–Backlund vector fields for the coupled non-linear Klein–Gordon system

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Abstract. We have analysed the types of non-linear coupled Klein-Gordon equations possessing non-trivial Lie-Backlund vector fields. The Dodd-Bullough equation occurs as a special case of our system. These systems have the special feature of Laving the lowest symmetry generator depending upon the fifth derivative of the fields.

Recently there have been various approaches for the study of completely integrable systems. One of the most effective and elegant methods is that of Lie-Backlund vector fields [1]. In this respect the case of coupled diffusion equations has been considered by Steeb [2]. Some important properties of non-trivial Lie-Backlund (LB) vector fields make it necessary to extend this analysis to other kinds of equations, which do not possess the lowest-order symmetries (those depending on U_1 and U_3). The first and foremost example of such an equation is the Dodd-Bullough [3] equation which does not possess any LB vector field starting with U_k with $k<5$. So although there is a mention of this equation in the paper by Steeb, the proper equation satisfied by the corresponding non-linear function occurring on the right-hand side of the Klein-Gordon equation was not deduced.

Here we begin with a more generalised setting by considering a pair of coupled non-linear Klein-Gordon (CNKG) equations for two fields u and ϑ:

$$\begin{aligned} u_{xt} &= f(u, \vartheta) \\ \vartheta_{xt} &= g(u, \vartheta). \end{aligned} \tag{1}$$

In the usual jet bundle approach we use the notation $u_x = u_1$, $u_{xx} = u_2$, $\vartheta_{xx} = \vartheta_2$, etc, and consider the submanifold given as

$$\begin{aligned} u_{1t} &= f(u, \vartheta) \\ \vartheta_{1t} &= g(u, \vartheta) \end{aligned} \tag{2}$$

and all other differential consequences of (2). That is, if we write

$$\begin{aligned} F &\equiv u_{1t} - f(u, \vartheta) = 0 \\ G &\equiv \vartheta_{1t} - g(u, \vartheta) = 0 \end{aligned}$$

then

$$F_1 \equiv u_{2t} - f_u u_1 - f_\vartheta \vartheta_1$$
$$F_2 \equiv u_{3t} - f_u u_2 - u_1^2 f_{uu} - 2u_1\vartheta_1 f_{u\vartheta} - \vartheta_2 f_\vartheta - \vartheta_1^2 f_{\vartheta\vartheta}$$
$$F_3 \equiv u_{4t} - f_u u_3 - 3u_1u_2 f_{uu} - u_1^3 f_{uuu} - \ldots$$

and

$$\begin{aligned} G_1 &\equiv \vartheta_{2t} - g_u u_1 - g_\vartheta \vartheta_1 \\ G_2 &\equiv \vartheta_{3t} - g_u u_2 - u_1^2 g_{uu} - 2u_1\vartheta_1 g_{u\vartheta} - \vartheta_2 g\vartheta - \vartheta_1^2 g_{\vartheta\vartheta} \\ G_3 &\equiv \vartheta_{4t} - u_3 g_u - \vartheta_3 g_\vartheta - 3u_1u_2 g_{uu} - 3\vartheta_1\vartheta_2 g_{\vartheta\vartheta} - u_1^3 g_{uuu} - \vartheta_1^3 g_{\vartheta\vartheta\vartheta} \\ &\quad + 3[u_1\vartheta_2 + u_2\vartheta_1] g_{u\vartheta} + 3u_1^2\vartheta_1 g_{uu\vartheta} + 3u_1\vartheta_1^2 g_{u\vartheta\vartheta}. \end{aligned} \tag{3}$$

For this coupled system of Klein-Gordon equations we assume that the lowest LB vector field starts with u_5 and all u_i with $i < 5$ so that

$$V = h(u_1 \ldots u_5, \vartheta_1 \ldots \vartheta_5)\frac{\partial}{\partial u} + k(u_1 \ldots u_5, \vartheta_1 \ldots \vartheta_5)\frac{\partial}{\partial \vartheta} \tag{4}$$

where we have tacitly assumed that the vector field does not depend on (u, ϑ) or on (x, t). It is usually observed that if at least one such LB vector field is admitted then a hierarchy exists. Furthermore, for application of (4) to equations (1) we require an extension of (4) involving u_t and ϑ_t. At this point it is not out of place to introduce the assumption that the functions h and k depend linearly on u_5 and ϑ_5. So we can write

$$\tilde{V} = h\begin{pmatrix} u_1 \ldots u_5 \\ \vartheta_1 \ldots \vartheta_5 \end{pmatrix}\frac{\partial}{\partial u} + k\begin{pmatrix} u_1 \ldots u_5 \\ \vartheta_1 \ldots \vartheta_5 \end{pmatrix}\frac{\partial}{\partial \vartheta} + D_t h\frac{\partial}{\partial u_t} + D_t k\frac{\partial}{\partial \vartheta_t} \tag{5}$$

where D_t is the operator defined by

$$D_t = \frac{\partial}{\partial t} + \sum u_{it}\frac{\partial}{\partial u_i} + \sum \vartheta_{it}\frac{\partial}{\partial \vartheta_i}. \tag{6}$$

Then the invariance requirement of (1) is expressed as

$$\mathscr{L}_\vartheta F \simeq 0 \qquad \mathscr{L}_\vartheta G \simeq 0 \tag{7}$$

where $\mathscr{L}_\vartheta$ denotes the Lie derivative with respect to the vector field V and $\simeq$ denotes the restriction to the solution manifold of (1). Finally

$$\begin{aligned} h &= a_1u_5 + a_2\vartheta_5 + h\begin{pmatrix} u_1 \ldots u_4 \\ \vartheta_1 \ldots \vartheta_4 \end{pmatrix} \\ k &= b_1u_5 + b_2\vartheta_5 + k\begin{pmatrix} u_1 \ldots u_4 \\ \vartheta_1 \ldots \vartheta_4 \end{pmatrix} \end{aligned} \tag{8}$$

where a_i, b_i are arbitrary constants. Equipped with these we now write out (7) in full, which reads

$$\sum u_{i+1}\Lambda\left(\frac{\partial h}{\partial u_i}\right) + \sum \vartheta_{i+1}\Lambda\left(\frac{\partial h}{\partial \vartheta_i}\right) + \sum u_{i+1t}\frac{\partial h}{\partial u_i} + \sum \vartheta_{i+1t}\frac{\partial h}{\partial \vartheta_i} = hf_u + kf_\vartheta \tag{9}$$

$$\sum u_{i+1}\Lambda\left(\frac{\partial k}{\partial u_i}\right) + \sum \vartheta_{i+1}\Lambda\left(\frac{\partial k}{\partial \vartheta_i}\right) + \sum u_{i+1t}\frac{\partial k}{\partial u_i} + \sum \vartheta_{i+1t}\frac{\partial k}{\partial \vartheta_i} = hg_u + kg_\vartheta \tag{10}$$

where Λ is the following operator:

$$\Lambda = \sum u_{it}\frac{\partial}{\partial u_i} + \sum \vartheta_{it}\frac{\partial}{\partial \vartheta_i}. \tag{11}$$

Equating coefficients of u_5 and ϑ_5, it is seen that h and k are independent of u_4 and ϑ_4 if the following conditions are satisfied:

$$a_2 = b_1 = 0 \qquad a_1 = b_2.$$

Separating again the coefficients of u_4 and ϑ_4, we observe

$$\begin{aligned} h &= u_3\xi_1 + \vartheta_3\xi_2 \\ k &= u_3\theta_1 + \vartheta_3\theta_2 \end{aligned} \tag{12}$$

with ξ_1, ξ_2 satisfying

$$\begin{aligned} &\Lambda\xi_1 + 5a_1u_1f_{uu} + 5a_1\vartheta_1 f_{u\vartheta} = 0 \\ &\Lambda\xi_2 + 5a_1\vartheta_1 f_{\vartheta\vartheta} + 5a_1u_1f_{u\vartheta} = 0 \\ &\Lambda\theta_1 + 5a_1u_1g_{uu} + 5a_1\vartheta_1 g_{u\vartheta} = 0 \\ &\Lambda\vartheta_2 + 5a_1\vartheta_1 g_{\vartheta\vartheta} + 5a_1u_1g_{u\vartheta} = 0 \end{aligned} \tag{13}$$

where the operator Λ reduces to

$$\Lambda = u_{2t}\frac{\partial}{\partial u_2} + \vartheta_{2t}\frac{\partial}{\partial \vartheta_2} + u_{1t}\frac{\partial}{\partial u_1} + \vartheta_{1t}\frac{\partial}{\partial \vartheta_1}. \tag{14}$$

Solving these sets of equations we obtain that ξ_i, θ_i $(i = 1, 2)$ possess the quadratic structure

$$\begin{aligned} \xi_1 &= \alpha u_2 + \beta\vartheta_2 + \sigma_1 u_1^2 + \sigma_2\vartheta_1^2 + \sigma_3 u_1\vartheta_1 \\ \xi_2 &= \alpha' u_2 + \beta'\vartheta_2 + \sigma_1' u_1^2 + \sigma_2'\vartheta_1^2 + \sigma_3' u_1\vartheta_1 \\ \theta_1 &= \gamma u_2 + \delta\vartheta_2 + \rho_1 u_1^2 + \rho_2\vartheta_1^2 + \rho_3 u_1\vartheta_1 \\ \theta_2 &= \gamma' u_2 + \delta'\vartheta_2 + \rho_1' u_1^2 + \rho_2'\vartheta_1^2 + \rho_3' u_1\vartheta_1. \end{aligned} \tag{15}$$

Simultaneously it is necessary that the functions f and g satisfy

$$\begin{aligned} &\alpha f_u + \beta g_u + 2\sigma_1 f + 5a_1 f_{uu} = 0 \\ &\alpha f_\vartheta + \beta g_\vartheta + 2\sigma_2 g + 5a_1 f_{u\vartheta} = 0 \end{aligned} \tag{16}$$

and similar equations from other sets. The consistency of these yield

$$\begin{aligned} \sigma_1\beta' - \sigma_1'\beta &= \rho_1\gamma' - \rho_1'\gamma \\ \sigma_2\beta' - \sigma_2'\beta &= \rho_2\gamma' - \rho_2'\gamma. \end{aligned} \tag{17}$$

So for coupled Klein-Gordon-like non-linear systems, when the functions f and g satisfy equations of the form (16), the symmetry generators have the form

$$\begin{aligned} \eta_1 = a_1u_5 &+ u_3[\alpha u_2 + \beta\vartheta_2 + \sigma_1 u_1^2 + \sigma_2\vartheta_1^2] \\ &+ \vartheta_3[\alpha' u_2 + \beta'\vartheta_2 + \sigma_1' u_1^2 + \sigma_2'\vartheta_1^2] + q(u_1\ \vartheta_1\ u_2\vartheta_2) \\ \eta_2 = b_2\vartheta_5 &+ u_3[\gamma u_2 + \delta\vartheta_2 + \rho_1 u_1^2 + \rho_2\vartheta_1^2] \\ &+ \vartheta_3[\gamma' u_2 + \delta'\vartheta_2 + \rho_1' u_1^2 + \rho_2^1\vartheta_1^2] + p(u_1\ \vartheta_1 u_2\vartheta_2). \end{aligned} \tag{18}$$

Substituting these forms of η_1 and η_2 and equating the coefficients of u_3, ϑ_3, the equations for q and p are obtained; in the form

$$\Lambda \frac{\partial p}{\partial u_2} = \text{a polynomial linear in } (u_2, \vartheta_2) \text{ and quadratic in } (u_1, \vartheta_1) \tag{19a}$$

$$\Lambda \frac{\partial p}{\partial \vartheta_2} = \text{a polynomial of the same structure as above.} \tag{19b}$$

We have not written the explicit structure of the right-hand sides of equations (19*a*) and (19*b*) as they are too elaborate and cumbersome, but the structure noted is good enough to suggest the form for p and q.

Actually we have set

$$\begin{aligned} q &= u_2^2 q_1(u_1, \vartheta_1) + \vartheta_2^2 q_2(u_1, \vartheta_1) + q'(u_1, \vartheta_1) \\ p &= u_2^2 p_1(u_1, \vartheta_1) + \vartheta_2^2 p_2(u_1, \vartheta_1) + p'(u_1, \vartheta_1) \end{aligned} \tag{20}$$

which leads to

$$2f\frac{\partial q_1}{\partial u_1} + 2g\frac{\partial q_1}{\partial \vartheta_1} + 2\sigma_1 f + 2\alpha f_u + 10a_1 f_{uu} - \gamma f_\vartheta + (\beta + 2\alpha')g_u = 0 \tag{21a}$$

$$2\sigma_1' f + \beta f_u + 10a_1 f_{u\vartheta} + (\alpha - \gamma')f_\vartheta + 2\beta' g_u + \alpha' g_\vartheta = 0 \tag{21b}$$

$$2\sigma_2 g + \beta' g_u + 10a_1 f_{u\vartheta} + (2\alpha - \delta)f_\vartheta + (\beta + \alpha')g_\vartheta = 0 \tag{21c}$$

$$2f\frac{\partial q_2}{\partial u_1} + 2g\frac{\partial q_2}{\partial \vartheta_1} + (2\beta + \alpha' - \delta')f_\vartheta + 10a_1 f_{\vartheta\vartheta} - \beta' f_u + 3\beta' g_\vartheta + 2\sigma_2' g = 0 \tag{21d}$$

and also

$$2f\frac{\partial p_1}{\partial u_1} + 2g\frac{\partial p_1}{\partial \vartheta_1} + (2\tau' + \delta - \alpha)g_u + 10b_2 g_{uu} - \gamma g_\vartheta + 3\gamma f_u + 2p_1 f = 0 \tag{22a}$$

$$2p_2 g + \gamma' g_\vartheta + 10b_2 g_{u\vartheta} + (\delta' - \beta)g_u + 2\gamma f_\vartheta + \delta f_u = 0 \tag{22b}$$

$$2p_1' f + \gamma f_\vartheta + 10b_2 g_{u\vartheta} + (2\delta' - \alpha')g_u + (\delta + \gamma')f_u = 0 \tag{22c}$$

$$2f\frac{\partial p_2}{\partial u_1} + 2g\frac{\partial p_2}{\partial \vartheta_1} + 2p_2' g + 2\delta' g_\vartheta + 10b_2 g_{\vartheta\vartheta} - \beta' g_u + (2\delta + \gamma')f_\vartheta = 0. \tag{22d}$$

It is interesting to observe that in the set of equations (21) there are two equations for the functions (q_1, q_2) occurring in the structure of the symmetries and two other coupled equations for the functions (f, g) giving the form of the non-linear equations. Similarly (p_1, p_2) are determined by equations (22*a*) and (22*d*), the other two equations of the set (22) again giving information for f and g. Now these equations suggest that

$$\begin{aligned} q_1 &= A_1(u, \vartheta)u_1 + B_1(u, \vartheta)\vartheta_1 \\ q_2 &= A_1'(u, \vartheta)u_1 + B_1'(u, \vartheta)\vartheta_1 \\ p_1 &= A_2(u, \vartheta)u_1 + B_2(u, \vartheta)\vartheta_1 \\ p_2 &= A_2'(u, \vartheta)u_1 + B_2'(u, \vartheta)\vartheta_1. \end{aligned} \tag{23}$$

A possible simplified solution is given as $B_1 = B_2 = 0$, $A_1 = \sigma_1$, $A_2' = 0$, etc. Finally, equating coefficients of u_2, ϑ_2 and their various powers we obtain equations for p' and

q' written as

$$\frac{\partial^2 q'}{\partial u_1^2} = \text{a polynomial of degree 3 in } u_1, \vartheta_1.$$

Similarly for p, and variations of q' and p' with respect to ϑ_1. So in general

$$q' = \sum_{i=0}^{5} \varepsilon_i u_1^{5-i} \vartheta_1^i \tag{24}$$

so that the explicit structure of the symmetry generators is

$$\begin{aligned}\eta_1 = a_1 u_5 + u_3(\alpha u_2 + \beta\vartheta_2 + \sigma_1 u_1^2 + \sigma_2 \vartheta_1^2) + v_3(\alpha' u_2 + \beta'\vartheta_2 + \sigma_1' u_1^2 + \sigma_2' \vartheta_1^2) \\ + u_2^2[\sigma_1 u_1 + \beta_1 \vartheta_1] + v_2^2[A_2 u_1 + B_2 \vartheta_1] + \sum_{i=0}^{5} \varepsilon_i u_1^{5-i} \vartheta_1^i\end{aligned} \tag{25}$$

with a similar form for η_2. At the same time we observe that the forms of the non-linear equations are determined by equations (21b), (21c), (22b) and (22c). By suitable adjustments of the arbitrary constants involved one can easily visualise that the solution admitted by these can be put in the form

$$\begin{aligned} g &= \mathrm{e}^{-u} \sinh 3\vartheta \\ f &= \mathrm{e}^{2u} - e^{-u} \cosh 3\vartheta. \end{aligned} \tag{26}$$

It is now interesting to observe that for $\vartheta = 0$, $g = 0$ and $f = \mathrm{e}^{2u} - e^{-u}$; so that the system reduces to the case of a Dodd–Bullough equation. In general, a system of the form (26) can be termed as a generalised Toda lattice equation and none of these possess any symmetry (non-trivial, other than space translation) with terms lower than u_5. Another solution for the system (21b), (21c) and (22b), (22c) is seen to be

$$\begin{aligned} f &= \mathrm{e}^{u} - \mathrm{e}^{-u} \cos \vartheta \\ g &= \mathrm{e}^{-u} \sin \vartheta \end{aligned} \tag{27}$$

which is the equation of the relativistic string. In the following we give a short discussion of the special case $\vartheta = 0$ for (27) to give some idea of the explicit computations involved.

In this case we have

$$\begin{aligned} u_{1t} &= f \\ D_x D_t \eta &= \eta f_u = \sum u_{i+1} \Lambda\left(\frac{\partial \eta}{\partial u_i}\right) + \sum u_{i+1} t \frac{\partial \eta}{\partial u_i} \end{aligned} \tag{28}$$

with

$$\eta = a_1 u_5 + h(u_4, \ldots, u_1).$$

Equation (28) yields $\Lambda(\partial h/\partial u_4) = 0$ implying independence of h on u_4. From the coefficient of u_4

$$\Lambda\left(\frac{\partial h}{\partial u_3}\right) + 5 a_1 u_1 f_{uu} = 0.$$

If we set

$$h = u_3 g(u_1, u_2)$$

leading to

$$u_{2t}\frac{\partial g}{\partial u_2}+u_{1t}\frac{\partial g}{\partial u_1}+5a_1u_1f_{uu}=0$$

with the choice

$$g=\alpha u_2+\beta u_1^2$$

we get

$$5a_1f_{uu}+\alpha f_u+2\beta f=0.$$

A solution for f is

$$f=\Lambda_1\,\mathrm{e}^{\sigma_1 u}+\Lambda_2\,\mathrm{e}^{\sigma_2 u}$$

with $\sigma_1\sigma_2$ being the roots of the equation

$$5a_1\sigma^2+\alpha\sigma+2\beta=0.$$

For $\sigma_2=-2\sigma_1$ we get the Dodd–Bullough equation and this leads to $\alpha^2=-5a_1\beta$, connecting the constants α and β. Such adjustment of constants can also be performed in the case of the coupled equations (26) and (27) for the reduction of the general equation to a specific case. In this particular case the form of the symmetry generator is found to be

$$\eta=a_1u_5+(\alpha u_2+\beta u_1^2)u_3+\sigma u_2^2u_1+\delta u_1^5.$$

In our discussions above we have used the machinery of Lie–Backlund vector fields to analyse a class of coupled Klein–Gordon equations, possessing no trivial generators starting with u_5. The relativistic string equation, generalised Toda lattice and Dodd-Bullough equation are well known members of this class. In each case the form of the symmetry generators are explicitly determined except for arbitrary constants. At this point we can mention that equations (26) and (27) were first considered by Fordy and Gibbons [4].

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